

Chapter 2

Deterministic Models in Quantum Notation

2.1 The Basic Structure of Deterministic Models

For deterministic models, we will be using the same Dirac notation. A physical state $|A\rangle$, where A may stand for any array of numbers, not necessarily integers or real numbers, is called an *ontological state* if it is a state our deterministic system can be in. These states themselves do not form a Hilbert space, since in a deterministic theory we have no superpositions, but we can declare that they form a *basis* for a Hilbert space that we may herewith define [102, 122], by deciding, once and for all, that all ontological states form an orthonormal set:

$$\langle A|B\rangle \equiv \delta_{AB}. \quad (2.1)$$

We can allow this set to generate a Hilbert space if we declare what we mean when we talk about superpositions. In Hilbert space, we now introduce the *quantum states* $|\psi\rangle$, as being more general than the ontological states:

$$|\psi\rangle = \sum_A \lambda_A |A\rangle, \quad \sum_A |\lambda_A|^2 \equiv 1. \quad (2.2)$$

A quantum state can be used as a *template* for doing physics. With this we mean the following:

A template is a quantum state of the form (2.2) describing a situation where the probability to find our system to be in the ontological state $|A\rangle$ is $|\lambda_A|^2$.

Note, that λ_A is allowed to be a complex or negative number, whereas the phase of λ_A plays no role whatsoever. In spite of this, complex numbers will turn out to be quite useful here, as we shall see. Using the *square* in Eq. (2.2) and in our definition above, is a fairly arbitrary choice; in principle, we could have used a different power. Here, we use the squares because this is by far the most useful choice; different powers would not affect the physics, but would result in unnecessary mathematical complications. The squares ensure that probability conservation amounts to a proper normalization of the template states, and enable the use of unitary matrices in our transformations.

Occasionally, we may allow the indicators A, B, \dots to represent continuous variables, a straightforward generalization. In that case, we have a continuous deterministic system; the Kronecker delta in Eq. (2.1) is then replaced by a Dirac delta, and the sums in Eq. (2.2) will be replaced by integrals. For now, to be explicit, we stick to a discrete notation.

We emphasise that the template states are not ontological. Hence we have *no* direct interpretation, as yet, for the inner products $\langle \psi_1 | \psi_2 \rangle$ if both $|\psi_1\rangle$ and $|\psi_2\rangle$ are template states. Only the absolute squares of $\langle A | \psi \rangle$, where $\langle A |$ is the conjugate of an ontological state, denote the probabilities $|\lambda_A|^2$. We briefly return to this in Sect. 5.5.3.

The time evolution of a deterministic model can now be written in operator form:

$$|A(t)\rangle = |P_{\text{op}}^{(t)} A(0)\rangle, \quad (2.3)$$

where $P_{\text{op}}^{(t)}$ is a permutation operator. We can write $P_{\text{op}}^{(t)}$ as a matrix $P_{AB}^{(t)}$ containing only ones and zeros. Then, Eq. (2.3) is written as a matrix equation,

$$|A(t)\rangle = U_{AB}(t) |B(0)\rangle, \quad U(t)_{AB} = P_{AB}^{(t)}. \quad (2.4)$$

By definition therefore, the matrix elements of the operator $U(t)$ in this bases can only be 0 or 1.

It is very important, at this stage, that we choose $P_{\text{op}}^{(t)}$ to be a genuine permutator, that is, it should be invertible.¹ If the evolution law is time-independent, we have

$$P_{\text{op}}^{(t)} = (P_{\text{op}}^{(\delta t)})^{t/\delta t}, \quad U_{\text{op}}(t) = (U_{\text{op}}(\delta t))^{t/\delta t}, \quad (2.5)$$

where the permutator $P_{\text{op}}^{(\delta t)}$, and its associated matrix $U_{\text{op}}(\delta t)$ describe the evolution over the shortest possible time step, δt .

Note, that no harm is done if some of the entries in the matrix $U(\delta t)_{ab}$, instead of 1, are chosen to be unimodular complex numbers. Usually, however, we see no reason to do so, since a simple rotation of an ontological state in the complex plane has no physical meaning, but it could be useful for doing mathematics (for example, in Sect. 15 of Part II, we use the entries ± 1 and 0 in our evolution operators).

We can now state our first important mathematical observation:

The quantum-, or template-, states $|\psi\rangle$ all obey the same evolution equation:

$$|\psi(t)\rangle = U_{\text{op}}(t) |\psi(0)\rangle. \quad (2.6)$$

It is easy to observe that, indeed, the probabilities $|\lambda_A|^2$ evolve as expected.²

Much of the work described in this book will be about writing the evolution operators $U_{\text{op}}(t)$ as exponentials: *Find a Hermitian operator H_{op} such that*

¹One can imagine deterministic models where $P_{\text{op}}^{(t)}$ does not have an inverse, which means that two different ontological states might both evolve into the same state later. We will consider this possibility later, see Chap. 7.

²At this stage of the theory, one may still define probabilities to be given as different functions of λ_A , in line with the observation just made after Eq. (2.2).

$$U_{\text{op}}(\delta t) = e^{-i H_{\text{op}} \delta t}, \quad \text{so that} \quad U_{\text{op}}(t) = e^{-i H_{\text{op}} t}. \quad (2.7)$$

This elevates the time variable t to be continuous, if it originally could only be an integer multiple of δt . Finding an example of such an operator is actually easy. If, for simplicity, we restrict ourselves to template states $|\psi\rangle$ that are orthogonal to the eigenstate of U_{op} with eigenvalue 1, then

$$H_{\text{op}} \delta t = \pi - i \sum_{n=1}^{\infty} \frac{1}{n} (U_{\text{op}}(n \delta t) - U_{\text{op}}(-n \delta t)) \quad (2.8)$$

is a solution of Eq. (2.7). This equation can be checked by Fourier analysis, see Part II, Sect. 12.2, Eqs. (12.8) – (12.10).

Note that a correction is needed: the lowest eigenstate $|\emptyset\rangle$ of H , the ground state, has $U_{\text{op}}|\emptyset\rangle = |\emptyset\rangle$ and $H_{\text{op}}|\emptyset\rangle = 0$, so that Eq. (2.8) is invalid for that state, but here this is a minor detail³ (it is the *only* state for which Eq. (2.8) fails). If we have a periodic automaton, the equation can be replaced by a finite sum, also valid for the lowest energy state, see Sect. 2.2.1.

There is one more reason why this is *not* always the Hamiltonian we want: its eigenvalues will always be between 0 and $2\pi/\delta t$, while sometimes we may want expressions for the energy that take larger values (see for instance Sect. 5.1).

We do conclude that there is always a Hamiltonian. We repeat that the ontological states, as well as all other template states (2.2) obey the Schrödinger equation,

$$\frac{d}{dt} |\psi(t)\rangle = -i H_{\text{op}} |\psi(t)\rangle, \quad (2.9)$$

which reproduces the discrete evolution law (2.7) at all times t that are integer multiples of δt . Therefore, we always reproduce *some kind* of “quantum” theory!

2.1.1 Operators: Beables, Changeables and Superimposables

We plan to distinguish three types of operators:

- (I) *beables*: these denote a property of the ontological states, so that beables are diagonal in the ontological basis $\{|A\rangle, |B\rangle, \dots\}$ of Hilbert space:

$$\mathcal{O}_{\text{op}}|A\rangle = \mathcal{O}(A)|A\rangle, \quad (\text{beable}). \quad (2.10)$$

- (II) *changeables*: operators that replace an ontological state by another ontological state, such as a permutation operator:

$$\mathcal{O}_{\text{op}}|A\rangle = |B\rangle, \quad (\text{changeable}); \quad (2.11)$$

These operators act as pure permutations.

³In Part II, we shall see the importance of having one state for which our identities fail, the so-called *edge state*.

(III) *superimposables*: these map ontological states onto superpositions of ontological states:

$$\mathcal{O}_{\text{op}}|A\rangle = \lambda_1|A\rangle + \lambda_2|B\rangle + \cdots. \quad (2.12)$$

Now, we will construct a number of examples. In Part II, we shall see more examples of constructions of beable operators (e.g. Sect. 15.2).

2.2 The Cogwheel Model

One of the simplest deterministic models is a system that can be in just 3 states, called (1), (2), and (3). The time evolution law is that, at the beat of a clock, (1) evolves into (2), (2) evolves into (3), and state (3) evolves into (1), see Fig. 2.1a. Let the clock beat with time intervals δt . As was explained in the previous section, we associate Dirac kets to these states, so we have the states $|1\rangle$, $|2\rangle$, and $|3\rangle$. The evolution operator $U_{\text{op}}(\delta t)$ is then the matrix

$$U_{\text{op}}(\delta t) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.13)$$

It is now useful to diagonalize this matrix. Its eigenstates are $|0\rangle_H$, $|1\rangle_H$, and $|2\rangle_H$, defined as

$$\begin{aligned} |0\rangle_H &= \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle), \\ |1\rangle_H &= \frac{1}{\sqrt{3}}(|1\rangle + e^{2\pi i/3}|2\rangle + e^{-2\pi i/3}|3\rangle), \\ |2\rangle_H &= \frac{1}{\sqrt{3}}(|1\rangle + e^{-2\pi i/3}|2\rangle + e^{2\pi i/3}|3\rangle), \end{aligned} \quad (2.14)$$

for which we have

$$U_{\text{op}}(\delta t) \begin{pmatrix} |0\rangle_H \\ |1\rangle_H \\ |2\rangle_H \end{pmatrix} = \begin{pmatrix} |0\rangle_H \\ e^{-2\pi i/3}|1\rangle_H \\ e^{-4\pi i/3}|2\rangle_H \end{pmatrix}. \quad (2.15)$$

In this basis, we can write this as

$$U_{\text{op}} = e^{-iH_{\text{op}}\delta t}, \quad \text{with} \quad H_{\text{op}} = \frac{2\pi}{3\delta t} \text{diag}(0, 1, 2). \quad (2.16)$$

At times t that are integer multiples of δt , we have, in this basis,

$$U_{\text{op}}(t) = e^{-iH_{\text{op}}t}, \quad (2.17)$$

Fig. 2.1 **a** Cogwheel model with three states. **b** Its three energy levels



but of course, this equation holds in every basis. In terms of the *ontological basis* of the original states $|1\rangle$, $|2\rangle$, and $|3\rangle$, the Hamiltonian (2.16) reads

$$H_{\text{op}} = \frac{2\pi}{3\delta t} \begin{pmatrix} 1 & \kappa & \kappa^* \\ \kappa^* & 1 & \kappa \\ \kappa & \kappa^* & 1 \end{pmatrix}, \quad \text{with } \kappa = -\frac{1}{2} + \frac{i\sqrt{3}}{6}, \quad \kappa^* = -\frac{1}{2} - \frac{i\sqrt{3}}{6}. \quad (2.18)$$

Thus, we conclude that a template state $|\psi\rangle = \lambda(t)|1\rangle + \mu(t)|2\rangle + \nu(t)|3\rangle$ that obeys the Schrödinger equation

$$\frac{d}{dt}|\psi\rangle = -iH_{\text{op}}|\psi\rangle, \quad (2.19)$$

with the Hamiltonian (2.18), will be in the state described by the cogwheel model at all times t that are an integral multiple of δt . This is enough reason to claim that the “quantum” model obeying this Schrödinger equation is *mathematically equivalent* to our deterministic cogwheel model.

The fact that the equivalence only holds at integer multiples of δt is not a restriction. Imagine δt to be as small as the Planck time, 10^{-43} seconds (see Chap. 6), then, if any observable changes take place only on much larger time scales, deviations from the ontological model will be unobservable. The fact that the ontological and the quantum model coincide at all integer multiples of the time dt , is physically important. Note, that the original ontological model was not at all defined at non-integer time; we could simply define it to be described by the quantum model at non-integer times.

The eigenvalues of the Hamiltonian are $\frac{2\pi}{3\delta t}n$, with $n = 0, 1, 2$, see Fig. 2.1b. This is reminiscent of an atom with spin one that undergoes a Zeeman splitting due to a homogeneous magnetic field. One may conclude that such an atom is actually a deterministic system with three states, or, a cogwheel, but only if the ‘proper’ basis has been identified.

The reader may remark that this is only true if, somehow, observations faster than the time scale δt are excluded. We can also rephrase this. To be precise, a Zeeman atom is a system that needs only 3 (or some other integer N) states to characterize it. These are the states it is in at three (or N) equally spaced moments in time. It returns to itself after the period $T = N\delta t$.

2.2.1 Generalizations of the Cogwheel Model: Cogwheels with N Teeth

The first generalization of the cogwheel model (Sect. 2.2) is the system that permutes N ‘ontological’ states $|n\rangle_{\text{ont}}$, with $n = 0, \dots, N-1$, and N some positive integer > 1 . Assume that the evolution law is that, at the beat of the clock,

$$|n\rangle_{\text{ont}} \rightarrow |n+1 \bmod N\rangle_{\text{ont}}. \quad (2.20)$$

This model can be regarded as the universal description of any system that is periodic with a period of N steps in time. The states in this evolution equation are

regarded as ‘ontological’ states. The model does not say anything about ontological states in between the integer time steps. We call this the *simple periodic cogwheel model* with period N .

As a generalization of what was done in the previous section, we perform a discrete Fourier transformation on these states:

$$|k\rangle_H \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{2\pi i k n / N} |n\rangle_{\text{ont}}, \quad k = 0, \dots, N-1; \quad (2.21)$$

$$|n\rangle_{\text{ont}} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-2\pi i k n / N} |k\rangle_H. \quad (2.22)$$

Normalizing the time step δt to one, we have

$$U_{\text{op}}(1)|k\rangle_H = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{2\pi i k n / N} |n+1 \bmod N\rangle_{\text{ont}} = e^{-2\pi i k / N} |k\rangle_H, \quad (2.23)$$

and we can conclude

$$U_{\text{op}}(1) = e^{-iH_{\text{op}}}; \quad H_{\text{op}}|k\rangle_H = \frac{2\pi k}{N}|k\rangle_H. \quad (2.24)$$

This Hamiltonian is restricted to have eigenvalues in the interval $[0, 2\pi)$. where the notation means that 0 is included while 2π is excluded. Actually, its definition implies that the Hamiltonian is periodic with period 2π , but in most cases we will treat it as being defined to be restricted to within the interval. The most interesting physical cases will be those where the time interval is very small, for instance close to the Planck time, so that the highest eigenvalues of the Hamiltonian will be so large that the corresponding eigen states may be considered unimportant in practice.

In the original, ontological basis, the matrix elements of the Hamiltonian are

$${}_{\text{ont}}\langle m|H_{\text{op}}|n\rangle_{\text{ont}} = \frac{2\pi}{N^2} \sum_{k=1}^{N-1} k e^{2\pi i k(m-n)/N}. \quad (2.25)$$

This sum can we worked out further to yield

$$H_{\text{op}} = \pi \left(1 - \frac{1}{N}\right) - \frac{\pi}{N} \sum_{n=1}^{N-1} \left(\frac{i}{\tan(\pi n / N)} + 1 \right) U_{\text{op}}(n). \quad (2.26)$$

Note that, unlike Eq. (2.8), this equation includes the corrections needed for the ground state. For the other energy eigenstates, one can check that Eq. (2.26) agrees with Eq. (2.8).

For later use, Eqs. (2.26) and (2.8), without the ground state correction for the case $U(t)|\psi\rangle = |\psi\rangle$, can be generalised to the form

$$H_{\text{op}} = C - \frac{\pi i}{T} \sum_{t_n > 0}^{t_n < T} \frac{U_{\text{op}}(t_n)}{\tan(\pi t_n / T)} \xrightarrow{T \rightarrow \infty} C - i \sum_{t_n \neq 0} \frac{U_{\text{op}}(t_n)}{t_n}, \quad (2.27)$$

where C is a (large) constant, T is the period, and $t_n = n\delta t$ is the set of times where the operator $U(t_n)$ is required to have some definite value. We note that this is a sum, not an integral, so when the time values are very dense, the Hamiltonian tends to become very large. There seems to be no simple continuum limit. Nevertheless, in Part II, we will attempt to construct a continuum limit, and see what happens (Sect. 13).

Again, if we impose the Schrödinger equation $\frac{d}{dt}|\psi\rangle_t = -iH_{\text{op}}|\psi\rangle_t$ and the boundary condition $|\psi\rangle_{t=0} = |n_0\rangle_{\text{ont}}$, then this state obeys the deterministic evolution law (2.20) at integer times t . If we take superpositions of the states $|n\rangle_{\text{ont}}$ with the Born rule interpretation of the complex coefficients, then the Schrödinger equation still correctly describes the evolution of these Born probabilities.

It is of interest to note that the energy spectrum (2.24) is frequently encountered in physics: it is the spectrum of an atom with total angular momentum $J = \frac{1}{2}(N-1)$ and magnetic moment μ in a weak magnetic field: the Zeeman atom. We observe that, after the discrete Fourier transformation (2.21), a Zeeman atom may be regarded as the simplest deterministic system that hops from one state to the next in discrete time intervals, visiting N states in total.

As in the Zeeman atom, we may consider the option of adding a finite, universal quantity δE to the Hamiltonian. It has the effect of rotating all states with the complex amplitude $e^{-i\delta E}$ after each time step. For a simple cogwheel, this might seem to be an innocuous modification, with no effect on the physics, but below we shall see that the effect of such an added constant might become quite significant later.

Note that, if we introduce any kind of perturbation on the Zeeman atom, causing the energy levels to be split in intervals that are no longer equal, it will no longer look like a cogwheel. Such systems will be a lot more difficult to describe in a deterministic theory; they must be seen as parts of a much more complex world.

2.2.2 The Most General Deterministic, Time Reversible, Finite Model

Generalizing the finite models discussed earlier in this chapter, consider now a model with a finite number of states, and an arbitrary time evolution law. Start with any state $|n_0\rangle_{\text{ont}}$, and follow how it evolves. After some finite number, say N_0 , of time steps, the system will be back at $|n_0\rangle_{\text{ont}}$. However, not all states $|n\rangle_{\text{ont}}$ may have been reached. So, if we start with any of the remaining states, say $|n_1\rangle_{\text{ont}}$, then a new series of states will be reached, and the periodicity might be a different number, N_1 . Continue until all existing states of the model have been reached. We conclude that the most general model will be described as a set of simple periodic cogwheel models with varying periodicities, but all working with the same universal time step δt , which we could normalize to one; see Fig. 2.2.

Fig. 2.2 Example of a more generic finite, deterministic, time reversible model

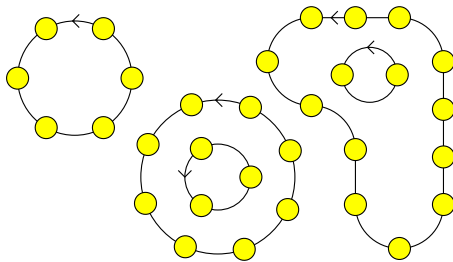


Figure 2.3 shows the energy levels of a simple periodic cogwheel model (left), a combination of simple periodic cogwheel models (middle), and the most general deterministic, time reversible, finite model (right). Note that we now shifted the energy levels of all cogwheels by different amounts δE_i . This is allowed because the index i , telling us which cogwheel we are in, is a conserved quantity; therefore these shifts have no physical effect. We do observe the drastic consequences however when we combine the spectra into one, see Fig. 2.3c.

Figure 2.3 clearly shows that the energy spectrum of a finite discrete deterministic model can quickly become quite complex.⁴ It raises the following question: given any kind of quantum system, whose energy spectrum can be computed. Would it be possible to identify a deterministic model that mimics the quantum model? To what extent would one have to sacrifice locality when doing this? Are there classes of deterministic theories that can be mapped on classes of quantum models? Which of these would be potentially interesting?

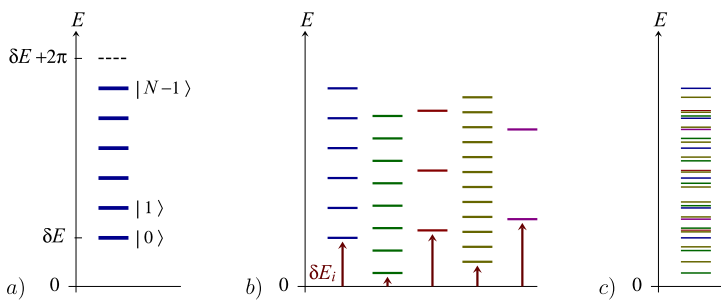


Fig. 2.3 **a** Energy spectrum of the simple periodic cogwheel model. δE is an arbitrary energy shift. **b** Energy spectrum of the model sketched in Fig. 2.2, where several simple cogwheel models are combined. Each individual cogwheel i may be shifted by an arbitrary amount δE_i . **c** Taking these energy levels together we get the spectrum of a generic finite model

⁴It should be self-evident that the models displayed in the figures, and subsequently discussed, are just simple examples; the real universe will be infinitely more complicated than these. One critic of our work was confused: “Why this model with 31 states? What’s so special about the number 31?” Nothing, of course, it is just an example to illustrate how the math works.

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